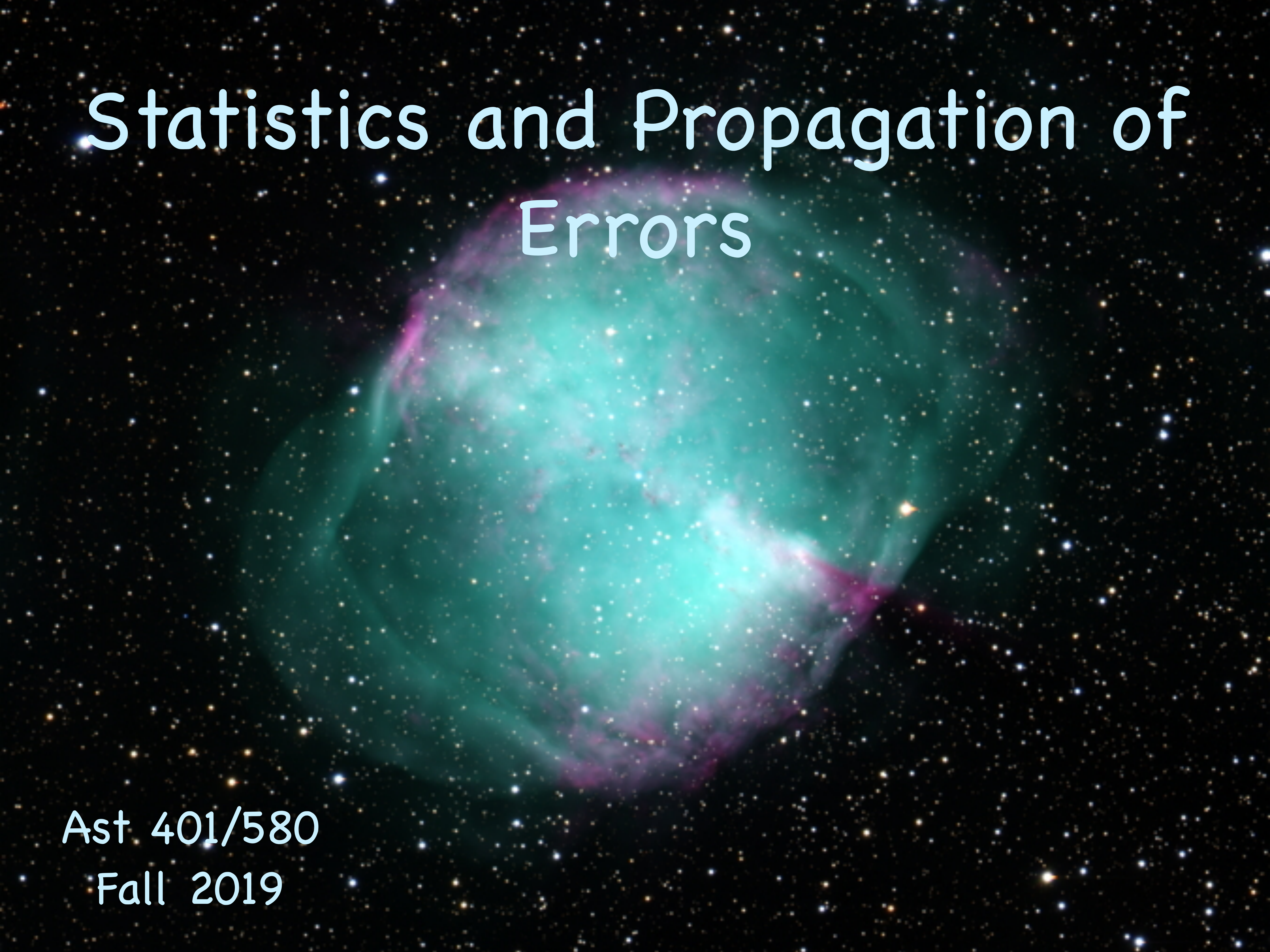


Statistics and Propagation of Errors

Ast 401/580

Fall 2019

Types of Errors

Consider measuring the amount of time it takes for a ball to drop from a table to the floor. There are basically two types of errors:

- Systematic errors
- Random errors

Random Errors

Normal (Gaussian) distribution

Consider many measurements of a quantity "x"

Probability function f

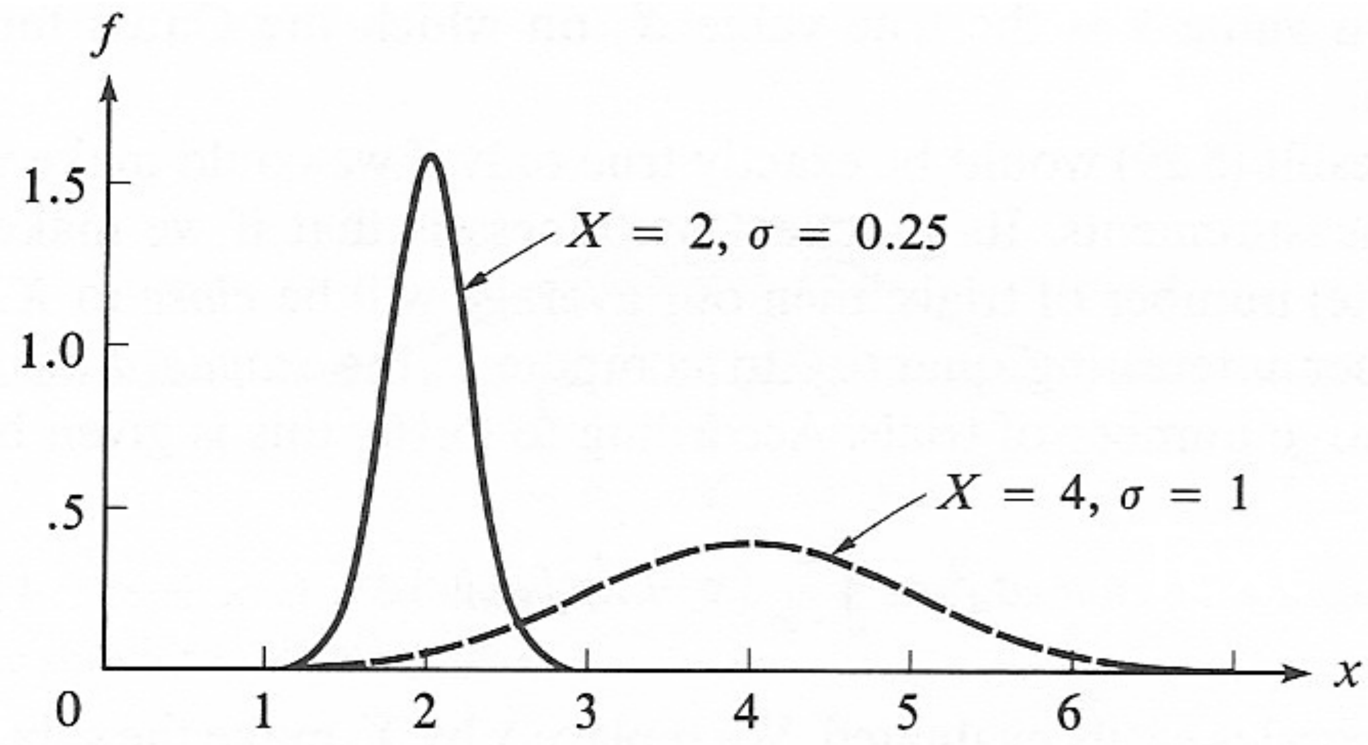


Figure 5.10. Two normal, or Gauss, distributions.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - X)^2}$$

Interpretation of the width parameter σ

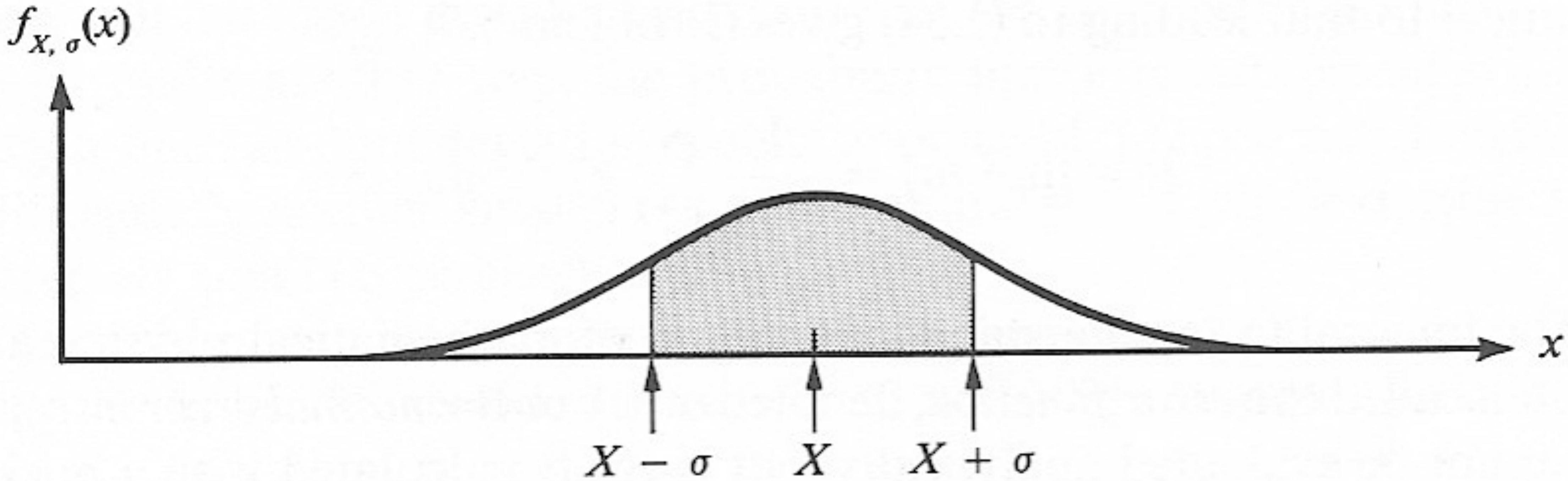


Figure 5.11. The shaded area between $X \pm \sigma$ is the probability of a measurement within one standard deviation of X.

$$P(within 1\sigma) = \int_{X-\sigma}^{X+\sigma} f_{X\sigma}(x)\,dx = 0.68$$

Large and Small σ

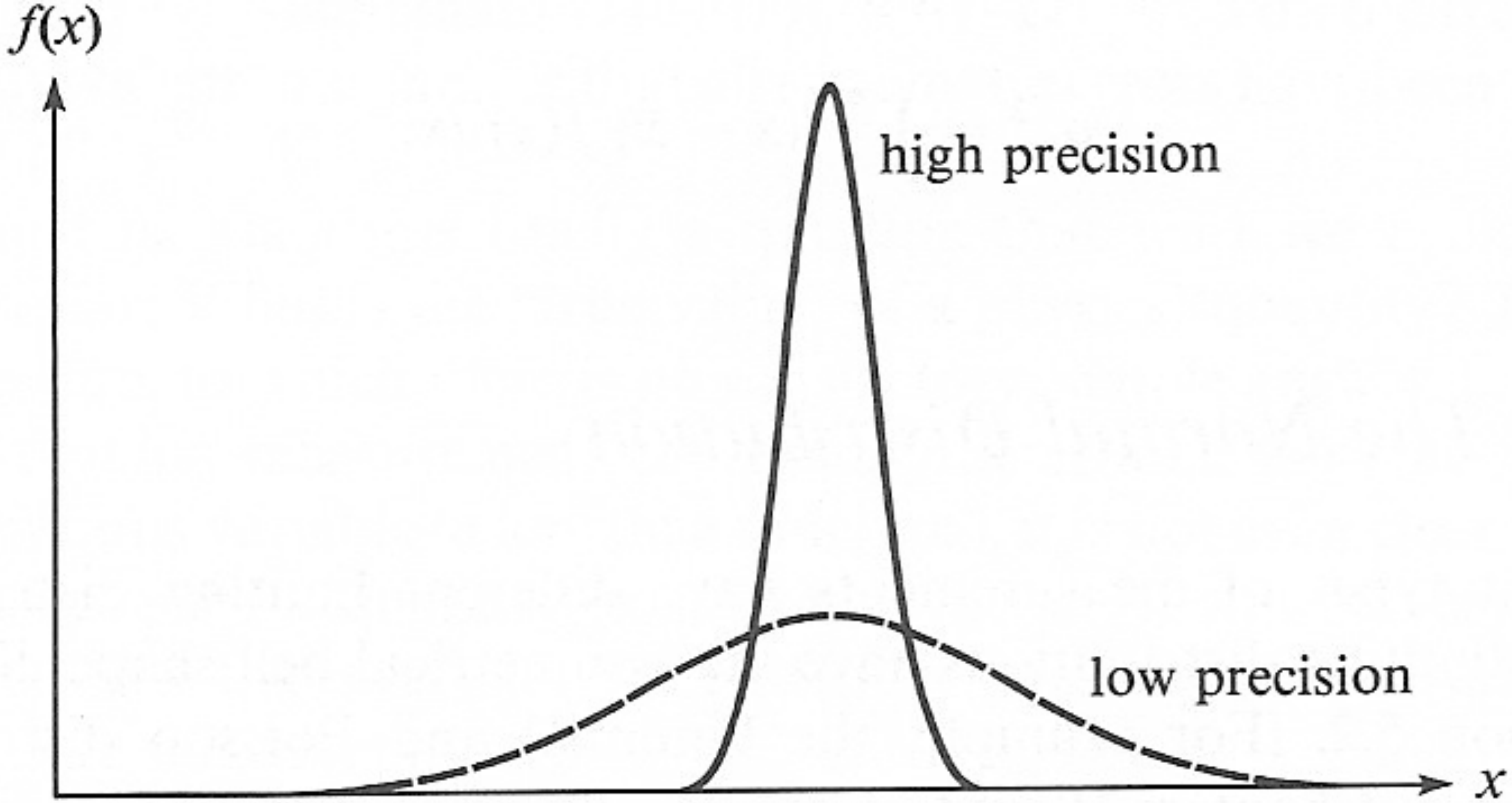


Figure 5.6. Two limiting distributions, one for a high precision measurement, the other for a low precision measurement.

Skipping to the chase

Let's call "σ" the standard deviation. Recall

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - X)^2}$$

We can also ask how uncertain is the mean:

$$\sigma_X = \frac{\sigma}{\sqrt{N}}$$

Poisson Distribution

Describes the results of experiments in which we we are counting things. (You can't have 0.25 of a photon....)

Poisson Distribution

$$P_\mu(\nu) = e^{-\mu}\frac{\mu^\nu}{\nu!}$$

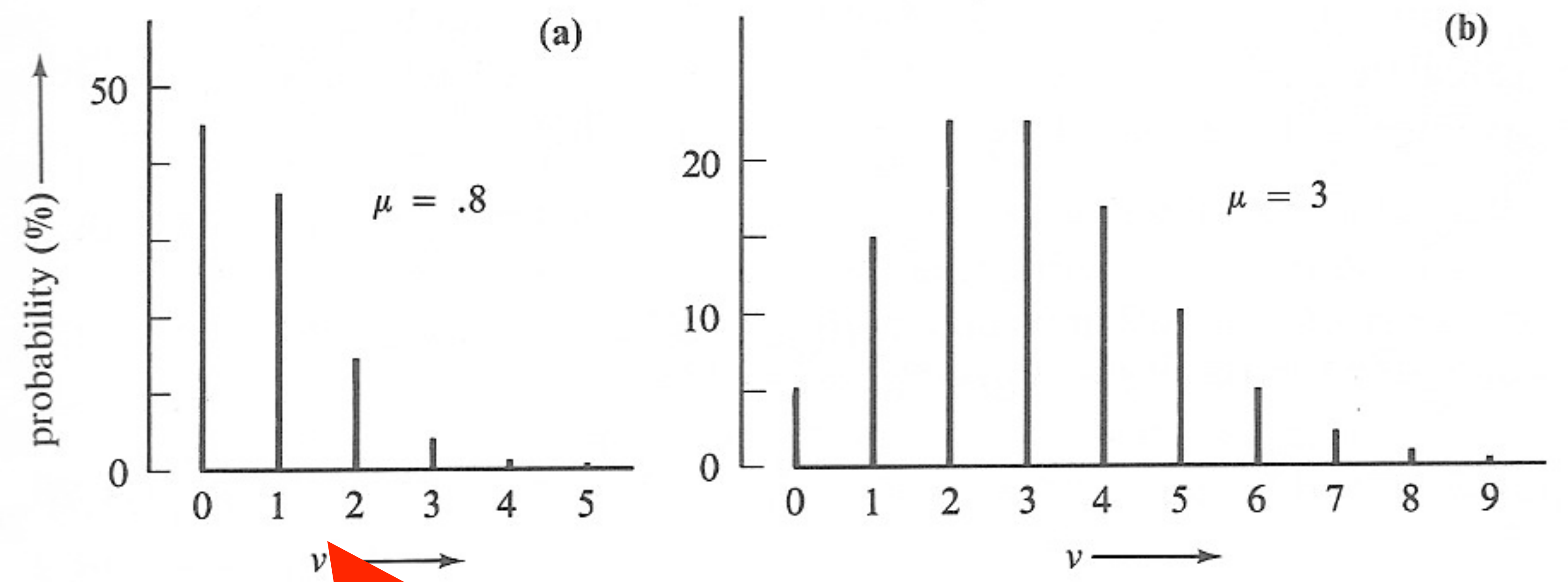


Figure 11.1. Poisson distributions with average counts $\mu = 0.8$ and 3.

$$P_{0.8}(0) = (0.45)\frac{(0.8)^0}{0!} = 0.45$$

$$P_{0.8}(1) = (0.45)\frac{(0.8)^1}{1!} = 0.36$$

$$P_{0.8}(2) = (0.45)\frac{(0.8)^2}{2!} = 0.14$$

Gauss

Poisson

$$f_{X\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-X)^2/2\sigma^2}$$

$$P_{\mu}(\nu) = e^{-\mu} \frac{\mu^{\nu}}{\nu!}$$

Continuous ‘x’

Two parameters: X, σ

Discrete ‘ν’

One parameter: μ

Important Property of Poisson Distribution

- Compute

$$\sigma_\nu^2 = \frac{1}{N}\sum_{i=1}^{N}(\nu_i - \bar{\nu})^2$$

$$\sigma_\nu^2 = \mu$$

$$\sigma_\nu = \sqrt{\mu}$$

- You make one measurement of ν counts in some time interval
- Your best answer and uncertainty are given by

$$\nu \pm \sqrt{\nu}$$

For "Large" μ Poisson ⟶ Gauss

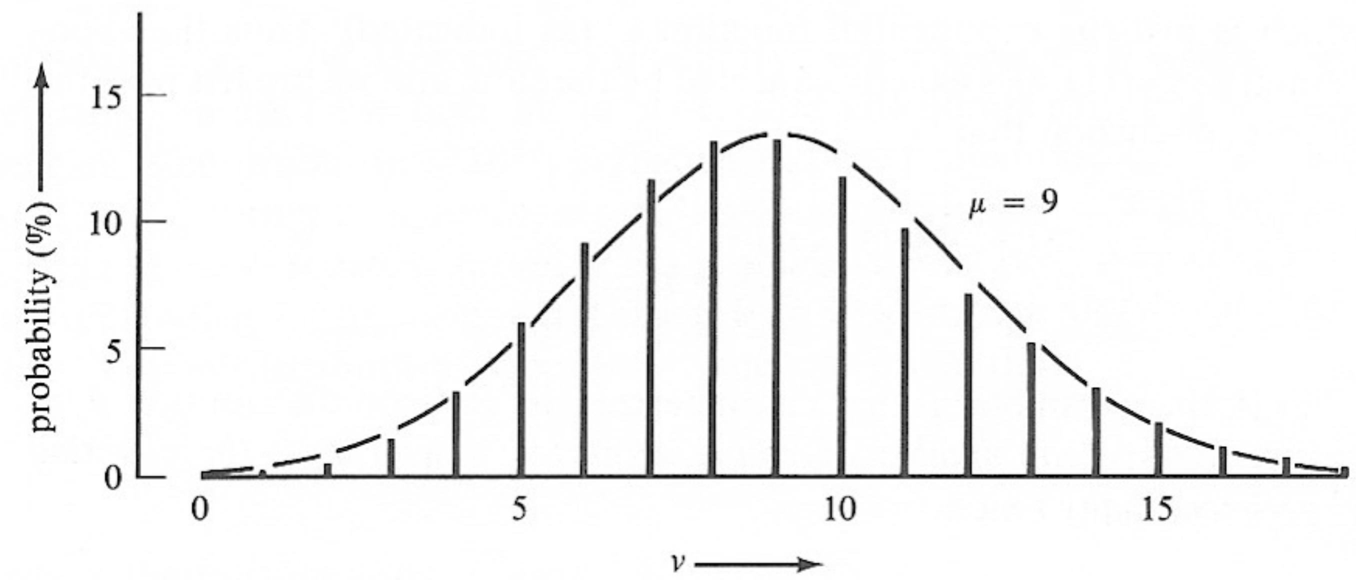


Figure 11.2. The Poisson distribution with $\mu = 9$. The broken curve is the Gauss distribution with the same center and standard deviation.

$$P_\mu(\nu) \cong f_{X,\sigma}(x)$$

$$X = \mu$$

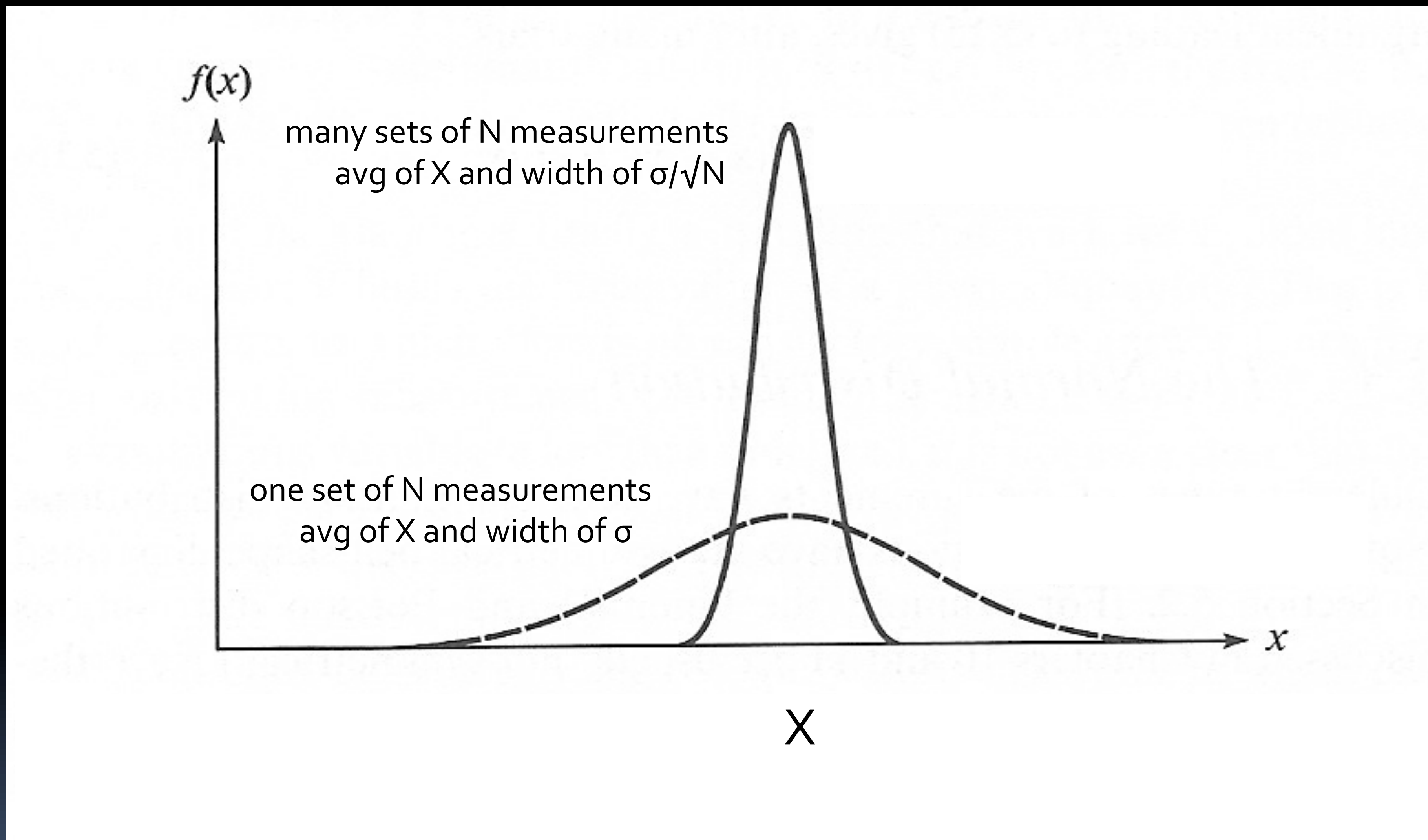
$$\sigma = \sqrt{\mu}$$

Poisson Distribution and Signal to Noise Ratio

- Remember $$\nu \pm \sqrt{\nu}$$
- Define

$$\frac{Signal}{Noise} = \frac{\nu}{\sqrt{\nu}} = \sqrt{\nu}$$

Standard Deviation of Mean



Imagine large number of experiments. In each, we make N measurements of x, and compute the average value. Our many average values will be normally distributed about X with width $\sigma/\sqrt{N}$

If we find the mean value of N measurements one time, then our best value and uncertainty is given by

$$\bar{x} \pm \frac{\sigma}{\sqrt{N}}$$

Uncertainties In A Sum

$$x \pm \delta_x \leftarrow \sigma / \sqrt{N}$$

$$y \pm \delta_y$$

We want the uncertainty in (x + y), and it is given by adding the individual uncertainties in quadrature

$$\sqrt{\delta_x^2 + \delta_y^2}$$

Assuming x and y are measured independently, and our errors are random in nature

Propagation of Errors

Imagine that you're making measurements using multiple "machines." The result of each machine is a, b, c, d... The end result of combining these all (in some way) is x, so that x is dependent on a, b, c, etc. Each time you "run the machine" you get a slightly different a, b, and c, because there's some error associated with each. What is the ultimate effect on x?

$$x = f(a,b,c\ldots)$$

$$(dx)^2 = (dx/da)^2 \ (da)^2 + (dx/db)^2 \ (db)^2 + (dx/dc)^2 (dc)^2$$

where (dx/da) is really the partial derivative of x wrt a.

Propagation of Errors

$(dx)^2 = (dx/da)^2 \; (da)^2 + (dx/db)^2 \; (db)^2 + (dx/dc)^2 \, (dc)^2$

Consider the simple case then that $x = a + b + c$. Take the partial derivatives:

$dx/da = 1$, $(dx/db)=1$, and $(dx/dc)=1$. You're left with:

$(dx)^2 = (da)^2 + (db)^2 + (dc)^2$

which is exactly the same as what we've been writing

as $\sigma^2_{tot} = \sigma^2_a + \sigma^2_b + \sigma^2_c$

Propagation of Errors

$(dx)^2 = (dx/da)^2 \ (da)^2 + (dx/db)^2 \ (db)^2 + (dx/dc)^2 \ (dc)^2$

What if instead $x = a\,(b/c)$? This is also straight-forward, depending upon how well you remember your Calculus!

In the end, the answer is simple enough:

$(\sigma_x/x)^2 = (\sigma_a/a)^2 + (\sigma_b/b)^2 + (\sigma_c/c)^2$